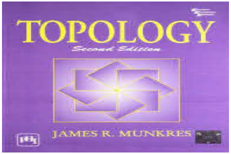
MMA 32 Topology



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Separation

Definition 1.

Let X be a topological space.

A **separation** of X is a pair U, V of disjoint nonempty open subsets of X

whose union is X.

(i.e., $X = U \cup V$ where $U \cap V = \phi$.)

Connected

Definition 2.

The space X is **connected** if there is no separation of X.

Note.

1. An alternative definition is that X is connected if and only if the only subsets of X that are both open and closed in X are the empty set and X itself.

2. Connectivity is a topological property.

Since connectivity is defined in term of open sets only.

If X is connected and Y is homeomorphic to X then Y is connected.

Lemma 1.

Let Y be a subspace of X.

A separation of Y is a pair of disjoint nonempty sets A and B

whose union is Y, neither of which contains limit point of the other.

The space Y is connected if there exists no separation of Y.

Proof.

Suppose that A and B form a separation of Y.

i.e., A and B are open in Y such that $A \cup B = Y$, $A \cap B = \phi$.

Since B is open in Y, $Y - B = A$ is closed in Y.

Then A is both open and closed in Y.

Let $\bar{A}$ denotes the closure of A in X.

By Theorem 17.4, the closure of A in Y is $\bar{A} \cap Y$.

Since A is closed in Y, $A = \bar{A} \cap Y$.

Since $A \cap B = \phi$ then $\bar{A} \cap B = \phi$.

By Theorem 17.6 $\bar{A}$ is the union of A and its limit points.

$\Rightarrow$ B contains no limit points of A.

Similarly, A contains no limit points of B.

Thus, a separation of Y is a pair of nonempty sets A and B

whose union is Y and neither contain a limit point of the other.

Conversely, suppose that A and B are disjoint nonempty sets whose

union is Y, neither of which contains a limit point of the other.

Then $\bar{A} \cap B = \phi$ and $A \cap \bar{B} = \phi$.

Since $A \cup B = Y$, then $(\bar{A} \cap Y) \cup (\bar{B} \cap Y) = Y$.

Since $A \cap (\bar{B} \cap Y) = \phi$ and $B \cap (\bar{A} \cap Y) = \phi$,

$\Rightarrow A = \bar{A} \cap Y$ and $B = \bar{B} \cap Y$.

Thus A and B are both closed in Y.

i.e., $A = Y - B$ and $B = Y - A$ are both open in Y.

Hence, A and B is a separation of Y.

Lemma 2.

If the sets C and D form a separation of X and

if Y is a connected subspace of X,

then Y lies entirely in either C or in D.

Proof. The sets C and D form a separation of X.

$\Rightarrow C$ and D are both open in X.

By def, the sets $C \cap Y$ and $D \cap Y$ are open in Y.

These two sets are disjoint and their union in Y.

Assume both are nonempty sets.

Then these two sets form a separation of Y,

CONTRADICTING the hypothesis that Y is connected.

$\Rightarrow$ Either $C \cap Y$ or $D \cap Y$ is an empty set.

Hence Y lies entirely in either C or in D.

Theorem 3.

The union of a collection of connected subspaces of X that have

a point in common is connected.

Proof.

Let $\{A_\alpha\}$ be a collection of subspaces of X.

Let p be a point in $\bigcap A_\alpha$.

To prove $Y = \bigcup A_\alpha$ is connected.

Assume that $Y = C \cup D$ where C and D are a separation of Y.

Point p must be in either C or D.

WLOG, say $p \in C$.

Since each A_α is connected.

By Lemma 2, it must lie entirely in either C or in D.

Since $p \in A_\alpha$, it cannot lie in D and hence $p \in C$.

Hence $A_\alpha \subset C$ for every α.

$\Rightarrow Y = \bigcup A_\alpha \subset C$.

It CONTRADICTS the fact that D is nonempty.

i.e., the assumption that there is a separation of Y is false.

Hence $Y = \bigcup A_\alpha$ is connected.

Theorem 4.

Let A be a connected subspace of X.

If $A \subset B \subset \bar{A}$, then B is also connected.

Proof. Let A be connected in X, and let $A \subset B \subset \bar{A}$.

To prove B is connected.

Assume that $B = C \cup D$ where C and D are a separation of B.

By Lemma 2, the set A must lie entirely in C or in D.

WLOG, suppose $A \subset C$.

Then $\bar{A} \subset \bar{C}$.

By Lemma 1, we have $\bar{C}$ and D are disjoint.

i.e., $\bar{C} \cap D = \phi$.

$\Rightarrow B \cap D = \phi$. $\quad (A \subset B \subset \bar{A} \subset \bar{C})$

This CONTRADICTS the fact that as part of a separation,

$D \neq \phi$ subset of B.

Thus the assumption that a separation of B exists is false.

Hence B is connected.

Theorem 5.

The image of a connected space under a continuous map is connected.

Proof.

Let $f : X \longrightarrow Y$ be a continuous function where X is connected.

Let $f(X) = Z$.

Since the map obtained from f by restricting its range to the space

Z is also continuous. (by Theorem 18.2(e))

Consider the case of a continuous surjective map $g : X \longrightarrow Z$.

ASSUME $Z = A \cup B$, where A and B form a separation of Z.

Since g is continuous, A and B are disjoint open sets.

$\Rightarrow g^{-1}(A)$ and $g^{-1}(B)$ are disjoint open sets, which are nonempty

whose union is X.

$\Rightarrow g^{-1}(A)$ and $g^{-1}(B)$ are a separation of X.

This CONTRADICTS the hypothesis that X is connected.

Hence there is no separation of $Z = f(X)$.

$\Rightarrow f(X)$ is connected.

Theorem 6.

A finite Cartesian product of connected spaces is connected.

Proof.

We prove the result for two connected spaces X and Y.

Then the general result follows by induction.

Choose $(a, b) \in X \times Y$.

Then the horizontal slice $X \times \{b\}$ is connected, being homeomorphic to X.

Similarly, the vertical slice $\{x\} \times Y$ is connected, being

homeomorphic to Y.

For each $x \in X$, define

$$T_x = (X \times \{b\}) \bigcup (\{x\} \times Y).$$

By Theorem 3, we have T_x is connected.

Next, consider $\bigcup_{x \in X} T_x = X \times Y$.

since the point (a, b) is common to each T_x.

By Theorem 3, the union is connected.

That is, $X \times Y$ is connected.

The proof for any finite product of connected spaces follows by induction.

Since $X_1 \times X_2 \times \ldots \times X_n$ is homeomorphic with

$(X_1 \times X_2 \times \ldots \times X_{n-1}) \times X_n$.

Definition 1.

A simply ordered set L having more than one element is a

linear continuum if the following hold:

(1) L has the least upper bound property

(i.e., every set with an upper bound has a least upper bound).

(2) If $x < y$, then there exists z such that $x < z < y$.

Connected Subspaces of the Real Line

Theorem 1.

If L is a linear continuum in the order topology,

then L is connected and so are intervals and rays in L.

Proof.

Recall that a subspace Y of L is convex if for every pair of points $a, b \in Y$

with $a < b$, then entire interval $[a, b] = \{x \in L \mid a \leq x \leq b\}$ lies in Y.

We prove that Y is a convex subspace of L, then Y is connected.

Let Y be convex.

Proof by contradiction.

Suppose assume that Y has a separation.

i.e., $Y = A \cup B$, where A and B are open in Y with $A \cap B = \phi$

Choose $a \in A$ and $b \in B$.

WLOG, say $a < b$.

Since Y is convex then $[a, b] \subset Y$.

Hence $[a, b]$ is the union of the disjoint sets $A_0 = A \cap [a, b]$ and

$B_0 = B \cap [a, b]$, each of which is open in $[a, b]$

(since A and B are open in Y)

in the subspace topology on $[a, b]$.

By Theorem 16.4, which is the same as the order topology .

Since $a \in A_0$ and $b \in B_0$, implies that $A_0 \neq \phi$ and $B_0 \neq \phi$.

So A_0 and B_0 form a separation of $[a, b]$.

Let $c = \sup A_0$.

We now show in to cases that $c \notin A_0$ and $c \notin B_0$, which

CONTRADICTS the fact that $[a, b] = A_0 \cup B_0$.

From this contradiction, it follows that Y is connected.

Case 1.

Suppose $c \in B_0$.

Then $c \neq a$ (since $a \in A$ and $A \cap B = \phi$).

So either $c = b$ or $a < c < b$.

In either case, since B_0 is open in $[a, b]$ then there is some interval

of the form $(d, c] \subset B_0$.

If $c = b$ we have a contradiction.

Since this implies that d is a smaller upper bound on A_0, but $d < c$.

If $c < b$ we note that $(c, d] \cap A_0 = \phi$ since c is an upper bound of A_0.

Then (with d as above where $(d, c] \subset B_0$) we have that

$$(d, b] = (d, c] \cup (c, b]$$

does not intersect A_0.

Again, d is a smaller upper bound on A_0 than c, a CONTRADICTION.

We conclude that $c \notin B_0$.

Case 2.

Suppose $c \in A_0$.

Then $c \neq b$ since $b \in B$.

So either $c = a$ or $a < c < b$.

Because A_0 is open in $[a, b]$, there must be some interval of the

form $[c, e)$ contained in A_0.

By property (2) of the linear continuum L,

we can choose a point $z \in L$ such that $c < z < e$.

Then $z \in A_0$, CONTRADICTING the fact that c is an upper bound of A_0.

We conclude that $c \notin A_0$.

We have shown that if Y is a convex subset of L then Y is connected.

Notice that intervals and rays are convex sets and so are connected.

Corollary 2. The real line R is connected and so are intervals and rays in R.

Theorem 3. **Intermediate Value Theorem.**

Let $f : X \to Y$ be a continuous map, where X is a connected space and Y is an ordered set in the order topology.

If a and b are two points of X and if r is a point of Y lying between $f(a)$ and $f(b)$, then there exists a point $c \in X$ such that $f(c) = r$.

Proof. Suppose f, X, and Y are as hypothesized.

The sets $A = f(X) \cap (-\infty, r)$ and $B = f(X) \cap (r, +\infty)$ are disjoint

and nonempty.

(since $(-\infty, r)$ and $(r, +\infty)$ are disjoint)

(since $f(a)$ is in one of these sets and $f(b)$ is in the other)

Each is open in $f(X)$ under the subspace topology.

ASSUME there is no point $c \in X$ such that $f(c) = r$.

Then $f(X) = A \cup B$ and A and B form a separation of $f(X)$.

Since X is connected and f is continuous

By Theorem 23.5, then $f(X)$ is connected, a CONTRADICTION.

So the assumption that there is no such $c \in X$ is false.

Hence $f(c) = r$ for some $c \in X$.

Definition 2.

Given points x and y of the space X, a path in X from x to y is a continuous map $f : [a, b] \to X$ such that $f(a) = x$ and $f(b) = y$.

A space X is path connected if every pair of points of X can be joined by a path in X.

Lemma A. If space X is path connected then it is connected.

Proof. Let X be path connected.

Proof by contradiction.

Suppose assume that X is not connected.

Then A and B form a separation of X.

Let $f : [a, b] \longrightarrow X$ be any path in X.

Since f is continuous and $[a, b]$ is a connected set in R.

By Theorem 23.5, $f([a, b])$ is connected in X.

So by Lemma 23.2, $f([a, b])$ lies either entirely in A or entirely in B.

But this cannot be the case if a is chosen from A and b is chosen from B,

a CONTRADICTION.

So our assumption that a separation of X exists is false.

Hence space X is connected.

Connected components of X

Definition 1.

Given a topological space X, define an equivalence relation on X by setting $x \sim y$ if there is a connected subspace of X containing both x and y.

The equivalence classes are called **components** or **connected components** of X.

Theorem 1.

The components of X are connected disjoint subspaces of X whose union is X, such that each nonempty connected subspace of X intersects only one of them.

Proof.

We know that any equivalence classes on a set partition the set.

Since the components are by definition equivalence classes.

Therefore, the components are disjoint non empty connected subspace of X whose union is all of X.

To prove each nonempty connected subspace of X intersects only one of them.

ASSUME connected subspace A of X intersects two disjoint nonempty components C_1 and C_2.

Say at x_1 and x_2, respectively.

Then $x_1 \sim x_2$ since $x_1, x_2 \in C_1$ and $x_1, x_2 \in C_2$.

Since the components are disjoint then $C_1 = C_2$, a CONTRADICTION.

So the assumption that a connected subspace can intersect

two components is false.

Thus, each nonempty connected subspace of X intersects

only one of them.

To show that a component C is connected.

Let $x_0 \in C$.

Then for each $x \in C$ we have $x_0 \sim x$, so there is a connected subspace A_x containing x_0 and x.

From the above, a connected subspace cannot intersect two components and so $A_x \subset C$.

Therefore, $C = \bigcup_{x \in C} A_x$.

Since each A_x is connected and $x_0 \in A_x$ for all $x \in C$

Then by Theorem 23.3, C is connected.

Definition 2.

We define another equivalence relation on the space X.

By defining $x \sim y$ if there is a path in X from x to y.

The equivalence classes are called **path components** of X.

Theorem 2.

The path components of X are path connected disjoint subspaces of X whose union is X, such that each nonempty path connected subspace of X intersects only one of them. (Proof similar to Theorem 1)

Lemma A.

Each connected component of a space X is closed.

If X has only finitely many connected components, then each component of X is also open.

Definition 3.

A space X is **locally connected at x**,

if for every neighborhood U of x, there is a connected neighborhood

V of x contained in U.

If X is locally connected at each of its points, it is said simply

to be **locally connected**.

Definition 4.

A space X is **locally path connected at x**,

if for every neighborhood U of x, there is a path-connected neighborhood

V of x contained in U.

If X is locally path connected at each of its points, then it is said

to be **locally path connected**.

Theorem 3.

A space X is locally connected if and only if for every open set U of X, each component of U is open in X.

Proof:

Suppose X is locally connected.

Let U be an open set in X.

Let $x \in C$ be a connected component of U.

Then by the definition of locally connected, there is a connected neighborhood V of x with $V \subset U$.

Since V is connected, by Theorem 25.1, it must lie entirely in the component C, $V \subset C$.

$\implies$ So C is open.

Conversely, suppose that the components of open sets in X are open.

Let $x \in X$ and let U be an arbitrary neighborhood of x.

Let C be the connected component of U which contains x.

Now C is connected and, by hypothesis, open in X.

So, by definition, X is locally connected.

Theorem 4.

A space X is locally path connected if and only if for every open set U of X, each path component of U is open in X.

(Proof similar to Theorem 3)

Theorem 5.

If X is a topological space, each path component of X lies in a component of X. If X is locally path connected, then the component and the path components are the same.

Proof:

Let C be a component of X.

Let $x \in C$.

Let P be the path component of X containing x.

By Lemma 24.A, P is connected.

By Theorem 25.1, $P \subset C$.

Suppose X is locally path connected.

ASSUME $P \neq C$.

Let Q denote the union of all the path components of X that are different

from P and which intersect C

(since $P \neq C$ then $Q \neq \phi$)

As above, by Lemma 24.A and Theorem 25.1, each of these path components must be in component C.

Since the path components partition X.

By Theorem 25.2, then the path components in Q, along with path component P, partition C

Thus, $C = P \cup Q$.

By hypothesized, X to be locally path connected.

By Theorem 25.4 each path component of X is open in X.

Therefore P (a path component of X) and

Q (a union of path components) are both open in X.

P and Q are disjoint by construction.

Thus P and Q form a separation of C, a CONTRADICTION.

So the assumption that $P \neq C$ is false.

Hence $P = C$.